

A remarkable representation of the Clifford group

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Abstract. The finite Heisenberg group knows when the dimension of Hilbert space is a square number. Remarkably, it then admits a representation such that the entire Clifford group—the automorphism group of the Heisenberg group—is represented by monomial phase-permutation matrices. This has a beneficial influence on the amount of calculation that must be done to find Symmetric Informationally Complete POVMs. I make some comments on the equations obeyed by the absolute values of the components of the SIC vectors, and on the fact that the representation partly suggests a preferred tensor product structure.

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THE HEISENBERG GROUP

The name "Heisenberg group" for the group we are interested in became widely used some 50 years after Heisenberg did the path breaking work that Arkady Plotnitsky describes elsewhere in this volume—and which is indeed closely related to this group. The very foundations of quantum mechanics are closely related to the Heisenberg group.

Actually there are many Heisenberg groups. They all admit a defining representation in terms of upper triangular matrices of the form

$$g(\phi, x, p) = \begin{pmatrix} 1 & x & \phi \\ 0 & 1 & p \\ 0 & 0 & 1 \end{pmatrix}. \quad (1)$$

Such matrices form a nilpotent group as soon as the entries x , p , and ϕ can be added and multiplied together in such a way that they form a group under addition. In other words, the entries must belong to a ring, but the ring can be chosen freely. If the entries are real numbers we obtain a three dimensional Lie group, whose Lie algebra includes the commutator between position and momentum—figuring so prominently in Heisenberg's discoveries. If the entries belong to the ring of integers modulo N we obtain a finite Heisenberg group, denoted $H(N)$, which has an essentially unique unitary irreducible representation in a Hilbert space of N dimensions. Other options are available, such as choosing the entries to belong to a finite field. This leads to finite groups represented in p^k -dimensional Hilbert spaces, where p is a prime number. Unless N is a prime number the latter groups differ from $H(N)$.

I have no intention to describe this zoo of groups in detail here. Suffice it to say that it played a major role in two of the most brilliant episodes of human thinking, nineteenth century geometry [1] and twentieth century physics—and that I believe they will play a major role in this century too. The focus in my talk was on the role they play in the problem of finding Symmetric Informationally Complete POVMs, or SICs for short. In the laboratory SICs correspond to a special kind of (doable [2]) measurements, of interest for the foundations of quantum mechanics [3]. My personal motivation for working on the problem will be revealed at the end.

A SIC is a collection of N^2 unit vectors in a Hilbert space of dimension N , such that they form a resolution of the identity, i.e. a POVM, and such that

$$|\langle \psi_I | \psi_J \rangle|^2 = \frac{1}{N+1} (1 + N \delta_{IJ}), \quad 1 \leq I, J \leq N^2. \quad (2)$$

Finding such collections of vectors is surprisingly hard. Zauner's conjecture states that they do exist for any N , that they can be chosen such that they form an orbit of the Heisenberg group $H(N)$, and such that every vector in the collection is left invariant under a special order three element of the automorphism group of $H(N)$ [4]. This automorphism group is what we call the Clifford group here. The evidence for Zauner's conjecture comes from explicit constructions, not from any structural understanding. Scott and Grassl give a wonderful summary of the status of this conjecture in the concluding section of their recent paper [5].

It is known that $H(N)$ is the only group that can serve the purpose if N is a prime not equal to 3 [6]. It is also known that the analogous problem concerning equiangular lines in real Hilbert spaces does not have a solution in general [7].

There is a problem with a similar flavour, which is that of finding a complete set of Mutually Unbiased orthonormal Bases, or MUB. Here it is believed that a solution exists only if the dimension equals a prime number or a power of a prime number. The existing solutions [8] are again closely related to the Heisenberg groups, but then one uses the version based on finite fields, which is not the same as $H(N)$ unless the dimension N is a prime number. It is odd that the SIC and MUB existence problems differ in this way. They also differ in that—at least from one point of view—the latter is partly understood: it is known that the existence of a complete set of MUB is equivalent to the existence of a unitary operator basis forming a flower with $N + 1$ petals, in the sense that the N^2 operators in the basis can be divided into $N + 1$ petals of mutually commuting operators, such that two different petals have only the unit element in common [9]. If we restrict ourselves to unitary operator bases whose elements form a group, this means that we must look for groups with a sufficient number of maximal abelian subgroups. It is known that no group can do better than the Heisenberg groups in this regard [10]. (Why one should restrict oneself to unitary operator bases of group type is not clear at all. But this is beside the point for the moment.)

SQUARE DIMENSIONS

It would seem as if $H(N)$, the Heisenberg group over the ring of integers modulo N , has a kind of "universal" structure that is insensitive to the choice of N , except for some special features that emerge when N is prime. However, closer inspection reveals that there is something very special about square dimensions too.

To see this, recall that the usual presentation of $H(N)$ is in terms of the root of unity

$$\omega = e^{\frac{2\pi i}{N}}, \quad (3)$$

and generators X and Z subject to the relations

$$ZX = \omega XZ, \quad X^N = Z^N = \mathbf{1}. \quad (4)$$

When presented like this the Heisenberg group admits a unique unitary representation—up to unitary equivalence [11]. The usual choice then is to use the representation where the cyclic subgroup generated by Z is diagonal. But suppose that the dimension $N = n^2$ is a square number. Then it is readily seen that

$$Z^n X^n = X^n Z^n. \quad (5)$$

This gives rise to a preferred maximal abelian subgroup all of whose N elements are of order n . What we [12] refer to as the phase permutation basis is the basis in which this special subgroup is diagonal. Its basis vectors are labelled $|r, s\rangle$, where r and s are integers modulo n , and we define the phase factor

$$q = e^{\frac{2\pi i}{n}} = \omega^n. \quad (6)$$

With the phase conventions we used the representation of the group generators is then

$$X|r, s\rangle = \begin{cases} |r, s+1\rangle & \text{if } s+1 \neq 0 \bmod n \\ q^r |r, 0\rangle & \text{if } s+1 = 0 \bmod n \end{cases} \quad (7)$$

$$Z|r, s\rangle = \omega^s |r-1, s\rangle. \quad (8)$$

This leads to

$$X^n |r, s\rangle = q^r |r, s\rangle, \quad Z^n |r, s\rangle = q^s |r, s\rangle. \quad (9)$$

All matrices here are monomial unitaries, also known as phase-permutation matrices, meaning that exactly one element in each column is non-zero, and equal to a phase factor, and similarly for the rows. Actually this much is true for the usual representation too. The reason why we refer to the new basis as the phase-permutation basis [12] is that its use has in its train that the entire Clifford group is given by phase-permutation matrices—and this is not at all true for the usual representation of the Heisenberg group. Recall that the Clifford group consists of all unitary operators U such

that $UX^iZ^jU^\dagger$ is again an element of the Heisenberg group—and that the Clifford group plays a very important role when one tries to understand the SIC problem, which is our main concern for the moment.

To show that the Clifford group is represented by phase-permutation matrices it is enough to observe that, being an automorphism group, it must permute the maximal abelian subgroups—known as stabiliser groups in quantum information theory—among themselves. But it must also preserve the order of all group elements. When the dimension equals n^2 there exists a unique stabiliser group consisting solely of elements of order n . This is the stabiliser group that determines the new basis. The Clifford group can only permute its elements among themselves, and the remarkable property of the basis follows immediately.

There is an interesting connection to the theory of theta functions here; in particular the ‘new’ representation is used by Mumford in his presentation of the theory of theta functions from the point of view of the Heisenberg group [13]. The Heisenberg group over the real numbers can be represented by generators S_x and T_y acting on the space of analytic functions through

$$(S_x f)(z) = f(z+x), \quad (T_y f)(z) = e^{\pi i y^2 \tau + 2\pi i y z} f(z+\tau y), \quad (10)$$

where τ is a complex number with positive imaginary part. If we restrict ourselves to the lattice of integer valued x and y there is a unique function invariant under these transformations, namely the theta function

$$\theta(z, \tau) = \sum_{k=-\infty}^{\infty} e^{\pi i k^2 \tau + 2\pi i k z}. \quad (11)$$

Using a net with somewhat finer meshes we catch an invariant n^2 -dimensional vector space of functions. This vector space admits a natural basis consisting of theta functions of rational characteristic. Thus we define a subgroup of the real Heisenberg group by insisting that na and nb are integers. Let $\theta_{0,0}(z) = \theta(z)$. As a basis for the vector space we take the functions

$$\theta_{a,b}(z) = S_b T_a \theta(z), \quad na, nb \in \mathbf{Z}. \quad (12)$$

The phase-permutation representation is recovered from the observations that

$$S_c \theta_{a,b} = \theta_{a,b+c}, \quad T_c \theta_{a,b} = e^{-2\pi i bc} \theta_{a+c,b}, \quad na, nb, nc \in \mathbf{Z} \quad (13)$$

and

$$\theta_{a+x,b+y} = e^{2\pi i ay} \theta_{a,b}, \quad x, y, na, nb \in \mathbf{Z}. \quad (14)$$

The reader may be familiar with the case $n = 1/2$, discussed in Whittaker and Watson [14]; in a notation closer to Jacobi’s we have $(\theta_{0,0}, \theta_{0,\frac{1}{2}}, \theta_{\frac{1}{2},0}, \theta_{\frac{1}{2},\frac{1}{2}}) = (\theta_3, \theta_4, \theta_2, \theta_1)$. Via elliptic curves the theta functions play a prominent role in the geometry of complex projective space, so the connection may prove helpful for the SIC problem, or in some other way—although at the moment this is a vague suggestion only.

In another direction, note the temptation to write $|r, s\rangle = |r\rangle \otimes |s\rangle$. Eq. (9) then says that the special subgroup we are basing the representation on is represented by local operators, that is by operators of the form $U_1 \otimes U_2$. Moreover the generator Z is a local operator—as indeed it was in the representation proposed by Weyl [11], where Z is diagonal. If then the basis vectors are separable one finds

$$Z = \text{diag}(1, \omega, \dots, \omega^{N-1}) = \text{diag}(1, q, \dots, q^n) \otimes \text{diag}(1, \omega, \dots, \omega^{n-1}). \quad (15)$$

The unitary that connects the two representations is a local operator with respect to this tensor product structure, so this property is preserved. We can conclude, for instance, that the n^4 vectors in a given SIC must fall in n multiplets with the same degree of entanglement in each. (Curiously, even more is true for some $N = 4$ SICs: all the 16 vectors have the same concurrence [15]. I do not understand why.) However, note that our two requirements—that $|r, s\rangle$ forms a phase-permutation basis and that $|r, s\rangle = |r\rangle \otimes |s\rangle$ —contains some ambiguities. The Main Property of the phase-permutation basis is that it leads to a representation of the automorphism group of the Heisenberg group by phase-permutation matrices, and this property is kept if the basis is changed by an arbitrary unitary phase-permutation matrix. In general this will disrupt the tensor product structure. Still the phase-permutation basis partly determines the tensor product structure of the Hilbert space, and this may turn out to be interesting—but again this is a somewhat vague idea at the moment.

THE SIC EQUATIONS FOR THE ABSOLUTE VALUES

I advertised that the phase-permutation basis simplifies the SIC problem to some extent. The main reason is that it is well adapted to exploit Zauner's conjecture, according to which any SIC vector is left invariant by a special element of the Clifford group. This does help, although not as much as one could hope for. The details can be found in the paper that we wrote [12]—and since the paper appeared very recently I cannot add much to the story here.

What I can add are a few comments that were left out of our paper. In a previous Växjö meeting Marcus Appleby discussed a subset of the SIC equations (2), which concern only the absolute values of the components of the SIC vectors [16]. He assumed that the SIC is an orbit of the Heisenberg group. In this case the SIC is completely determined by a single fiducial vector $|\psi_0\rangle$ such that

$$|\langle\psi_0|X^iZ^j|\psi_0\rangle|^2 = \begin{cases} 1 & i = j = 0 \\ \frac{1}{N+1} & \text{otherwise} . \end{cases} \quad (16)$$

The equations involve quartic polynomials in N variables, and most known solutions have been derived using Gröbner bases and dedicated computer programs such as MAGMA [5]. Still it is interesting to manipulate them a bit further, in order to see what is involved. Let the components of the fiducial vector be given by

$$z_a = \sqrt{p_a} e^{i\mu_a} . \quad (17)$$

In the usual representation it is known that the SIC equations are equivalent to the equations [17, 18]

$$\sum_{a=0}^{N-1} p_a^2 = \frac{2}{N+1} , \quad (18)$$

$$\sum_{a=0}^{N-1} p_a p_{a+x} = \frac{1}{N+1} , \quad x \in \{1, 2, \dots, N-1\} , \quad (19)$$

$$\sum_{a=0}^{N-1} \bar{z}_a \bar{z}_{a+k-i} z_{a+k} z_{a-i} = 0 , \quad i, k \neq 0 . \quad (20)$$

Interestingly there are N equations that do not involve the phases at all. Moreover these equations have an interesting geometrical interpretation, arising when we project all vectors onto the simplex spanned by the basis vectors in a cross section of the convex body of density matrices. The equations imply that the image of the N^2 SIC vectors is a regular simplex of a characteristic size, inscribed in the larger simplex [16]. Unfortunately the geometric interpretation also suggests that the absolute values p_i cannot be determined by these equations alone, since the orientation of the image simplex is not determined.

If we examine eqs. (18-19) we see that they remain the same if the sign of the integer x is changed. Hence there are at most $k+1$ equations independent equations, regardless of whether $N = 2k$ or $N = 2k+1$. We also see that they imply

$$\left(\sum_{a=0}^{N-1} p_a \right)^2 = 1 . \quad (21)$$

So normalisation is included—as in fact it was from the start, before the SIC equations were brought to this form. If $N = 2k$ we can also derive that

$$\left(\sum_{r=0}^{k-1} p_{2r} - \sum_{r=0}^{k-1} p_{2r+1} \right)^2 = \frac{1}{N+1} , \quad (22)$$

which again is easily imposed, and means that the equations can be linearised if $N = 2$. However, I cannot find any further relation that strikes the eye and is easy to handle.

Very similar equations can be deduced in the phase permutation basis, where the absolute values are naturally denoted by $\sqrt{p_{rs}}$. Recall that the dimension is $N = n^2$. From the equations for a Heisenberg covariant SIC one deduces that [12]

$$\sum_{r,s=0}^{n-1} p_{rs} p_{rs} = \frac{2}{N+1} , \quad (23)$$

$$\sum_{r,s=0}^{n-1} p_{rs} p_{r+x,s+y} = \frac{1}{N+1} . \quad (24)$$

Here x, y are integers modulo n , not both zero. In itself this is a slight improvement on the previous result; the number of independent equations goes up to $k+2$ if $N = 2k$. This means that the absolute values are determined by these equations alone if $N = 4$.

Let $N = 4$. Then there are four equations for the absolute values, all of which can be brought to a form analogous to eqs. (21-22):

$$\left\{ \begin{array}{l} p_{00}^2 + p_{01}^2 + p_{10}^2 + p_{11}^2 = \frac{2}{5} \\ 2p_{00}p_{01} + 2p_{10}p_{11} = \frac{1}{5} \\ 2p_{00}p_{10} + 2p_{01}p_{11} = \frac{1}{5} \\ 2p_{00}p_{11} + 2p_{01}p_{10} = \frac{1}{5} \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} (p_{00} + p_{01} + p_{10} + p_{11})^2 = 1 \\ (p_{00} + p_{01} - p_{10} - p_{11})^2 = \frac{1}{5} \\ (p_{00} + p_{10} - p_{01} - p_{11})^2 = \frac{1}{5} \\ (p_{00} + p_{11} - p_{01} - p_{10})^2 = \frac{1}{5} \end{array} \right. . \quad (25)$$

After taking the square roots we obtain a linear system of equations. Since the components appear symmetrically in the equations we may assume without loss of generality that $p_{00} \geq p_{01} \geq p_{10} \geq p_{11} \geq 0$. This determines three of the signs. If the sign of the fourth square root is positive one finds that

$$p_{01} = p_{10} = p_{11} = \frac{5 - \sqrt{5}}{20} . \quad (26)$$

The negative sign is inconsistent with $p_{rs} \geq 0$. Hence the moduli are completely determined, and solving for the phases afterwards is not too hard. Notice that the calculation is not only straightforward, it also gives every Heisenberg covariant SIC in four dimensions, with no input from numerical searches.

The phase-permutation basis comes into its own when we impose Zauner's conjecture. By some further permutations and rephasings of the basis vectors one can block diagonalise a Zauner unitary, so that it consists of three by three blocks of the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , \quad (27)$$

together with some diagonal elements. Each such block contributes three eigenvalues $(1, e^{2\pi i/3}, e^{4\pi i/3})$ to the spectrum. Given that the spectrum of a Zauner unitary is known [4] one can deduce that there will be 3 diagonal elements if $N = 3l$, and only one if $N = 3l + 1$. Regardless of whether the block diagonalisation is carried through, one concludes that an invariant vector will have $l + 1$ independent components only, where $N = n^2 = 3l$ or $3l + 1$. The resulting equations (23-24) for the absolute values are then easy to manage, at least for $N = 9, 16, 25$ which we did explicitly. But it turns out that the number of independent equations is too small to determine all the absolute values—and solving the equations that involve the phases is not easy at all [12]. In fact it has never been done, in any basis, for $N = 16, 25$.¹

CONCLUSION

The phase-permutation basis partly suggests a preferred tensor product structure $\mathcal{H}^n \otimes \mathcal{H}^n$ of the Hilbert space \mathcal{H}^N . The commuting operators X^n and Z^n are local operators. This is interesting in itself. It also has the wonderful

¹ This is what I said in my talk. Eventually things worked out better than I expected—the final version of our paper does contain the solution for $N = 16$ [12].

consequence that the entire Clifford group is represented by monomial unitary matrices. This simplifies the SIC existence problem, but more ideas are needed to trivialise it in dimensions $N > 4$. Perhaps some can be found.

All of this reminds me of Fermat's Last Theorem. The statement of Fermat's theorem is very simple. In itself it appears to be a curiosity only, but in the course of proving it mathematicians were led to develop deep theories that changed the face of their subject. I like to think that the SIC existence problem has a similar status within quantum mechanics.

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